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Representation of Completely Positive Maps Between Partial *-Algebras

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A characterization of the invariant completely positive conjugate-bilinear maps from an arbitrary partial *-algebra to a semiassociative, locally convex partial *-algebra is given. The result generalizes Stinespring's characterization of completely positive maps on C^*-algebras, as well as its recent extensions by a number of authors.

1. INTRODUCTION

We consider the class $CP(\mathscr{A},\mathscr{B})$, of completely positive conjugate-bilinear maps from an arbitrary partial *-algebra $\mathscr{A}$ to a semiassociative, locally convex partial *-algebra $\mathscr{B}$. Our main result is the characterization of the subclass $ICP(\mathscr{A},\mathscr{B})$, consisting of the members of $CP(\mathscr{A},\mathscr{B})$ that are invariant in some sense. The result generalizes the well-known Stinespring characterization (Stinespring, 1955) of completely positive maps on C^*-algebras, as well as its recent extensions by Powers (1974), Lassner and Lassner (1977), Paschke (1973), and Ekhaguere and Odiobala (1991), and is formulated in terms of locally convex partial *-algebraic modules. These are generalizations of inner product modules over B*-algebras (Paschke, 1973) and are introduced in Section 2. As the general theory of locally convex partial *-algebraic modules is of independent interest, these modules will be exclusively studied elsewhere. The characterization of the members of $ICP(\mathscr{A}, \mathscr{B})$ is undertaken in Section 3. As a by-product, it is seen there that there is a profuse supply of locally convex partial *-algebraic modules, since each member of $CP(\mathscr{A},\mathscr{B})$ gives rise to such a module.

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In the rest of this section, we outline some of the fundamental notions employed in the sequel.

Let $\mathcal{A}$ be a *partial *-algebra* (Antoine *et al.*, 1990, 1991; Antoine and Inoue, 1990; Ekhaguere, 1988, 1993) with involution # and partial multiplication $\circ$.

If $x \in \mathcal{A}$, we write $R(x)$ [resp. $L(x)$] for the set of *right* [resp. *left*] *multipliers* of x. The set $M(x) = L(x) \cap R(x)$ consists of the *universal multipliers* of x. More generally, if $\mathcal{C} \subseteq \mathcal{A}$, we shall use the following notation:

$$R(\mathcal{C}) = \bigcap_{x \in \mathcal{C}} R(x) = \text{universal right multipliers of } \mathcal{C}$$

$$L(\mathcal{C}) = \bigcap_{x \in \mathcal{C}} L(x) = \text{universal left multipliers of } \mathcal{C}$$

$$M(\mathcal{C}) = L(\mathcal{C}) \cap R(\mathcal{C}) = \text{universal multipliers of } \mathcal{C}$$

$\mathcal{A}$ is called *semiassociative* iff $x, y \in \mathcal{A}$, with $y \in R(x)$, implies $y \circ z \in R(x)$ and $x \circ (y \circ z) = (x \circ y) \circ z$ for all $z \in R(\mathcal{A})$. Under these conditions, we shall often write $x \circ (y \circ z)$ or $(x \circ y) \circ z$ simply as $x \circ y \circ z$.

We remark that when $\mathcal{A}$ is semiassociative, then both $L(\mathcal{A})$ and $R(\mathcal{A})$ are algebras [but, in general, not *-algebras, since $L(\mathcal{A})^{\#} = R(\mathcal{A})$ and $R(\mathcal{A})^{\#} = L(\mathcal{A})$, where $\mathcal{C}^{\#} = \{x^{\#}: x \in \mathcal{C}\}$, for $\mathcal{C} \subseteq \mathcal{A}$] and $M(\mathcal{A})$ is a *-algebra.

A member e of $\mathcal{A}$ is called a *unit* (and $\mathcal{A}$ is then said to be *unital*) iff $e \in M(\mathcal{A})$, $e^{\#} = e$, and $x \circ e = x = e \circ x$, for all $x \in \mathcal{A}$. A unit of a unital partial *-algebra is unique.

The *positive cone* of $\mathcal{A}$ is the set $\mathcal{A}_+$ given by

$$\mathcal{A}_+ = \{x_1^{\#} \circ x_1 + \cdots + x_n^{\#} \circ x_n : x_1, x_2, \ldots, x_n \in R(\mathcal{A}), n \in \mathbb{N}\}$$

We say that $x \in \mathcal{A}$ is *positive* if $x \in \mathcal{A}_+$ and write $x \geq 0$.

Given a Hausdorff locally convex topology τ on $\mathcal{A}$, we call the pair $(\mathcal{A}, \tau)$ a *locally convex partial *-algebra* iff the following properties are satisfied:

- $(\mathcal{A}_0, \tau)$ is a Hausdorff locally convex space, where $\mathcal{A}_0$ is the underlying linear space of $\mathcal{A}$.
- The map $x \mapsto x^{\#}$ of $\mathcal{A}$ into $\mathcal{A}$ is τ-continuous.
- The map $x \mapsto z \circ x$ of $\mathcal{A}$ into $\mathcal{A}$ is τ-continuous for all $z \in L(\mathcal{A})$.
- And/or the map $x \mapsto x \circ z$ of $\mathcal{A}$ into $\mathcal{A}$ is τ-continuous for all $z \in R(\mathcal{A})$.

2. PARTIAL *-ALGEBRAIC MODULES

In this section, $(\mathcal{B}, \tau_{\mathcal{B}})$ is a locally convex partial *-algebra (with involution * and partial multiplication written as juxtaposition), whose topology $\tau_{\mathcal{B}}$ is generated by a family $\{|\cdot|_\alpha : \alpha \in \Delta\}$ of seminorms, and D is a linear space which is also a right $R(\mathcal{B})$-module in the sense that $x.a + y.b \in D$, whenever $x, y \in D$ and $a, b \in R(\mathcal{B})$, where the action of $R(\mathcal{B})$ on D is written as $z.c$ for $z \in D$, $c \in R(\mathcal{B})$.

Definition 2.1. A *$\mathcal{B}$-valued inner product* on D is a conjugate-bilinear map $\langle\cdot,\cdot\rangle_{\mathcal{B}}: D \times D \to \mathcal{B}$ such that:

(i) $\langle x, x\rangle_{\mathcal{B}} \in \mathcal{B}_+$, $\forall x \in D$, and $\langle x, x\rangle_{\mathcal{B}} = 0$ only if $x = 0$.
(ii) $\langle x, y\rangle_{\mathcal{B}} = \langle y, x\rangle_{\mathcal{B}}^*$, $\forall x, y \in D$.
(iii) $\langle x, y.b\rangle_{\mathcal{B}} = \langle x, y\rangle_{\mathcal{B}} b$, $\forall x, y \in D$, $b \in R(\mathcal{B})$.

Notation. If $\langle\cdot,\cdot\rangle_{\mathcal{B}}$ is a $\mathcal{B}$-valued inner product on D, define $\|\cdot\|_\alpha: D \to [0, \infty)$ by

$$\|x\|_\alpha = \sqrt{|\langle x, x\rangle_{\mathcal{B}}|_\alpha}, \qquad x \in D, \quad \alpha \in \Delta$$

Then, the following inequality holds:

$$\tfrac{1}{2}\left(|\langle x, y\rangle_{\mathcal{B}}|_\alpha + |\langle y, x\rangle_{\mathcal{B}}|_\alpha\right) \leq \|x\|_\alpha \|y\|_\alpha \tag{*}$$

for all $x, y \in D$, $\alpha \in \Delta$. It follows that $\|\cdot\|_\alpha$ is a *seminorm* on D for each $\alpha \in \Delta$.

We write $\tau_{D,\mathcal{B}}$ for the locally convex topology on D generated by the family $\{\|\cdot\|_\alpha: \alpha \in \Delta\}$.

Remarks. 1. If $|\cdot|_\alpha$ is *-invariant, i.e., if $|a^*|_\alpha = |a|_\alpha$, $\forall a \in \mathcal{B}$, $\alpha \in \Delta$, then the inequality (*) reduces to

$$|\langle x, y\rangle_{\mathcal{B}}|_\alpha \leq \|x\|_\alpha \|y\|_\alpha, \qquad \forall x, y \in D, \quad \alpha \in \Delta$$

2. In addition to properties (i)–(iii) of Definition (2.1), we make the following assumption about the action of $R(\mathcal{B})$ on D:

(iv) For each $b \in R(\mathcal{B})$, the map

$$l_R(b): \quad (D, \tau_{D,\mathcal{B}}) \to (D, \tau_{D,\mathcal{B}})$$

given by

$$l_R(b)x = x.b, \qquad x \in D$$

is *continuous*.

Definition 2.2. A triple $(D, \langle\cdot,\cdot\rangle_{\mathcal{B}}, \tau_{D,\mathcal{B}})$ for which (i)–(iv) of Definition 2.1 hold will be called a *locally convex $(\mathcal{B}, \tau_{\mathcal{B}})$-module.*

Remarks. 1. A locally convex $(\mathcal{B}, \tau_{\mathcal{B}})$-module is a generalization of an inner product module over a B^*-algebra (Paschke, 1973).

2. If D is already $\tau_{D,\mathcal{B}}$-complete, then the triple $(D, \langle\cdot,\cdot\rangle_{\mathcal{B}}, \tau_{D,\mathcal{B}})$ will be called a *complete* locally convex $(\mathcal{B}, \tau_{\mathcal{B}})$-module; in case D is not $\tau_{D,\mathcal{B}}$-

complete, one passes to a complete locally convex $(\mathcal{B}, \tau_{\mathcal{B}})$-module by completion.

Notation. Let $(D, \langle\cdot,\cdot\rangle_{\mathcal{B}}, \tau_{D,\mathcal{B}})$ be a locally convex $(\mathcal{B}, \tau_{\mathcal{B}})$-module and $(X, \tau_{X,\mathcal{B}})$ the $\tau_{D,\mathcal{B}}$-completion of D. The $\mathcal{B}$-valued inner product $\langle\cdot,\cdot\rangle_{\mathcal{B}}$ on D extends to a $\mathcal{B}$-valued inner product, denoted again by $\langle\cdot,\cdot\rangle_{\mathcal{B}}$, on X, and the triple $(X, \langle\cdot,\cdot\rangle_{\mathcal{B}}, \tau_{X,\mathcal{B}})$ is a complete locally convex $(\mathcal{B}, \tau_{\mathcal{B}})$-module.

We write $L(D, X)$ for the linear space of all continuous linear maps from $(D, \tau_{D,\mathcal{B}})$ to $(X, \tau_{X,\mathcal{B}})$.

Examples. 1. Let $(\mathcal{B}, \tau_{\mathcal{B}})$ be a locally convex partial *-algebra whose topology is generated by the family $\{|\cdot|_\alpha\colon \alpha \in \Delta\}$ of seminorms. Take $D \equiv R(\mathcal{B})$; define $\langle\cdot,\cdot\rangle_{\mathcal{B}}\colon D \times D \to \mathcal{B}$ by $\langle x, y\rangle_{\mathcal{B}} = x^*y$, $x, y \in D$, and $\tau_{D,\mathcal{B}}$ as the locally convex topology on D generated by the family $\{\|\cdot\|_\alpha\colon \alpha \in \Delta\}$ given by $\|x\|_\alpha = (|\langle x, x\rangle_{\mathcal{B}}|_\alpha)^{1/2}$. Then $(D, \langle\cdot,\cdot\rangle_{\mathcal{B}}, \tau_{D,\mathcal{B}})$ is a locally convex $(\mathcal{B}, \tau_{\mathcal{B}})$-module.

2. Let $\mathcal{H}$ be a pre-Hilbert space with inner product $\langle\cdot,\cdot\rangle_{\mathcal{H}}$ and $(\mathcal{B}, \tau_{\mathcal{B}})$ a locally convex partial *-algebra, as in Example 1 above. Take D as the algebraic tensor product $D \equiv R(\mathcal{B}) \otimes \mathcal{H}$ and define $\langle\cdot,\cdot\rangle_{\mathcal{B}}\colon D \times D \to \mathcal{B}$ by

$$\langle a \otimes x, b \otimes y\rangle_{\mathcal{B}} = \langle x, y\rangle_{\mathcal{H}} a^*b, \qquad a, b \in R(\mathcal{B}), \quad x, y \in \mathcal{H}$$

Letting $\tau_{D,\mathcal{B}}$ be the locally convex topology on D whose family of seminorms $\{\|\cdot\|_\alpha\colon \alpha \in \Delta\}$ is given by

$$\|a \otimes x\|_\alpha = \sqrt{|\langle a \otimes x, a \otimes x\rangle_{\mathcal{B}}|_\alpha}$$

Then the triple $(D, \langle\cdot,\cdot\rangle_{\mathcal{B}}, \tau_{D,\mathcal{B}})$ is a locally convex $(\mathcal{B}, \tau_{\mathcal{B}})$-module.

Definition 2.3. Let $(D, \langle\cdot,\cdot\rangle_{\mathcal{B}}, \tau_{D,\mathcal{B}})$ be a locally convex $(\mathcal{B}, \tau_{\mathcal{B}})$-module and $(X, \tau_{X,\mathcal{B}})$ the $\tau_{D,\mathcal{B}}$-completion of D. A map $T \in L(D, X)$ will be called a *module map* iff

$$T(x.b) = (Tx).b, \qquad \forall x \in D, \quad b \in R(\mathcal{B})$$

Notation. 1. We write $L_{\text{mod}}(D, X)$ for the set of all $T \in L(D, X)$ to which correspond some T^*, with domain in X containing D and forming a right $R(\mathcal{B})$-module, such that

$$\langle Tx, y\rangle_{\mathcal{B}} = \langle x, T^*y\rangle_{\mathcal{B}}, \qquad \forall x, y \in D$$

The map T^* will be called an *adjoint* of T. It is clear that an adjoint of a map in $L(D, X)$ is unique, as D is dense in X. Hence, $L_{\text{mod}}(D, X)$ is a *-invariant linear space which is, in general, not an algebra.

2. The linear space $L_{\text{mod}}(D, X)$ may, however, be given the structure of a *partial *-algebra* by specifying a partial multiplication $\circ$ and an involution $^+$ on it as follows. For $T \in L_{\text{mod}}(D, X)$, define T^+ by

$$T^+ \equiv T^* |_D$$

Then with $\Gamma \subseteq L_{\text{mod}}(D, X)$ given by

$$\Gamma = \{(T_1, T_2) \in (L_{\text{mod}}(D, X))^2:$$
$$T_2 D \subseteq \text{domain of } T_1^{+*} \text{ and } T_1^+ D \subseteq \text{domain of } T_2^*\}$$

we define $T_1 \circ T_2$ by

$$T_1 \circ T_2 \equiv T_1^{+*} T_2$$

whenever $(T_1, T_2) \in \Gamma$. We work with the partial *-algebra $(L_{\text{mod}}(D, X), {}^+, \circ)$ in the sequel and often denote it simply by $L^+_{\text{mod}}(D, X)$.

Proposition 2.4. Every member of $L^+_{\text{mod}}(D, X)$ is a module map.

Definition 2.5. A module *-representation of a partial *-algebra $\mathcal{A}$ (with involution $^\#$ and partial multiplication $\cdot$) is a map π from $\mathcal{A}$ into some $L^+_{\text{mod}}(D, X)$ such that the following hold on D:

(i) $\pi(\lambda_1 a + \lambda_2 b) = \lambda_1 \pi(a) + \lambda_2 \pi(b)$, $\forall \lambda_1, \lambda_2 \in \mathbb{C}$, $a, b \in \mathcal{A}$.
(ii) $\pi(a^\#) = \pi(a)^+$, $\forall a \in \mathcal{A}$.
(iii) If $A \in L(b)$, then $\pi(a) \in L(\pi(b))$ and

$$\pi(a \cdot b) = \pi(a) \circ \pi(b)$$

Remark. Since $\pi(a)$ is in $L^+_{\text{mod}}(D, X)$ for each $a \in \mathcal{A}$, it follows from Proposition 2.4 that $\pi(a)(x.b) = (\pi(a)x).b$ for all $a \in \mathcal{A}$, $b \in R(\mathcal{B})$.

3. COMPLETE POSITIVITY

Let $\mathcal{A}$ be a partial *-algebra, $(\mathcal{B}, \tau_{\mathcal{B}})$ a *semiassociative* locally convex partial *-algebra (with involution * and partial multiplication written as juxtaposition), whose topology τ_β is generated by a family $\{|\cdot|_\alpha : \alpha \in \Delta\}$ of seminorms, and $\text{Bil}(\mathcal{A}, \mathcal{B})$ the set of all maps $\varphi: \mathcal{A} \times \mathcal{A} \to \mathcal{B}$ with the properties

(i) $\varphi(x, \lambda_1 y + \lambda_2 z) = \lambda_1 \varphi(x, y) + \lambda_2 \varphi(x, z)$
(ii) $\varphi(x, y)^* = \varphi(y, x)$

for all $x, y, z \in \mathcal{A}$, $\lambda_1, \lambda_2 \in \mathbb{C}$. The members of $\text{Bil}(\mathcal{A}, \mathcal{B})$ are therefore *conjugate-bilinear* maps.

Notation. In the sequel, the map $\langle\langle \cdot, \cdot \rangle\rangle: R(\mathcal{B}) \times \mathcal{B} \to \mathcal{B}$ is defined by

$$\langle\langle a, b \rangle\rangle = a^* b, \qquad a \in R(\mathcal{B}), \quad b \in \mathcal{B}$$

Remark. Let n be a positive integer. A member of $\mathrm{Bil}(\mathscr{A}, \mathscr{B})$ is called *n-positive* iff

$$\sum_{j,k=1}^{n} \langle\langle b_j, \varphi(a_j, a_k)b_k\rangle\rangle = \sum_{j,k=1}^{n} b_j^* \varphi(a_j, a_k)b_k) \geq 0$$

$$\forall a_1, a_2, \ldots, a_n \in \mathscr{A}, \quad b_1, b_2, \ldots, b_n \in R(\mathscr{B})$$

The *completely positive* maps from $\mathscr{A} \times \mathscr{A}$ to $\mathscr{B}$ form a subset of $\mathrm{Bil}(\mathscr{A}, \mathscr{B})$ and are defined as follows.

Definition 3.1. $\varphi \in \mathrm{Bil}(\mathscr{A}, \mathscr{B})$ is called *completely positive* if φ is n-positive for each n.

Notation. Denote the set of all completely positive members of $\mathrm{Bil}(\mathscr{A}, \mathscr{B})$ by $CP(\mathscr{A}, \mathscr{B})$.

Remark. 1. Every $\varphi \in CP(\mathscr{A}, \mathscr{B})$ is automatically positive in the sense that $\varphi(x, x) \geq 0 \ \forall x \in \mathscr{A}$.

2. If $(D, \langle\cdot,\cdot\rangle_{\mathscr{B}}, \tau_{D,\mathscr{B}})$ is a locally convex $(\mathscr{B}, \tau_{\mathscr{B}})$-module, $(X, \tau_{X,\mathscr{B}})$ the completion of $(D, \tau_{D,\mathscr{B}})$, π a module *-representation of $\mathscr{A}$ in $L^+_{\mathrm{mod}}(D, X)$, and x_0 some fixed member of D, then the map $\varphi\colon \mathscr{A} \times \mathscr{A} \to \mathscr{B}$ defined by

$$\varphi(a, b) = \langle \pi(a)x_0, \pi(b)x_0\rangle_{\mathscr{B}}, \qquad a, b \in \mathscr{A} \tag{$*$}$$

is *completely positive.* It is not known, without any restriction on $\mathscr{A}$, if every $\varphi \in CP(\mathscr{A}, \mathscr{B})$ is always of the form (*). We shall characterize a subset of $CP(\mathscr{A}, \mathscr{B})$ whose members have representations of the form (*).

3. Form the algebraic tensor product $\mathscr{A} \otimes \mathscr{B}$. This is a right $R(\mathscr{B})$-module if we define the module action by

$$(a \otimes b).c = a \otimes (bc)$$

$\forall a \in \mathscr{A}$, $b \in \mathscr{B}$, and $c \in R(\mathscr{B})$.

Proposition 3.2. To each $\varphi \in CP(\mathscr{A}, \mathscr{B})$ there corresponds a locally convex $(\mathscr{B}, \tau_{\mathscr{B}})$-module.

Proof. Let $\varphi \in CP(\mathscr{A}, \mathscr{B})$. Define

$$\langle\cdot,\cdot\rangle_{\varphi,\mathscr{B}}\colon \quad (\mathscr{A} \otimes \mathscr{B}) \times (\mathscr{A} \otimes \mathscr{B}) \to \mathscr{B}$$

by

$$\left\langle \sum_{j=1}^{n} a_j \otimes b_j, \sum_{k=1}^{m} \alpha_k \otimes \beta_k \right\rangle_{\varphi,\mathscr{B}} = \sum_{j=1}^{n}\sum_{k=1}^{m} \langle\langle b_j, \varphi(a_j, \alpha_k)\beta_k\rangle\rangle$$

for $a_1, \ldots, a_n$, $\alpha_1, \ldots, \alpha_m$ in $\mathscr{A}$ and $b_1, \ldots, b_n$, $\beta_1, \ldots, \beta_m$ in $\mathscr{B}$. Then, $\langle\cdot,\cdot\rangle_{\varphi,\mathscr{B}}$ is a $\mathscr{B}$-valued inner product on $\mathscr{A} \otimes \mathscr{B}$. Let $N_\varphi = \{x \in \mathscr{A} \otimes \mathscr{B}$:

$\langle x, x\rangle_{\varphi,\mathcal{B}} = 0\}$. One checks that N_φ is a right $R(\mathcal{B})$-submodule of $\mathcal{A} \otimes \mathcal{B}$. Denote $(\mathcal{A} \otimes \mathcal{B})\backslash N_\varphi$ by X_φ^0 and let $\lambda_\varphi(x)$ be the coset of X_φ^0 containing $x \in \mathcal{A} \otimes \mathcal{B}$. A $\mathcal{B}$-valued inner product, denoted again by $\langle\cdot,\cdot\rangle_{\varphi,\mathcal{B}}$, is induced in a natural way by $\langle\cdot,\cdot\rangle_{\varphi,\mathcal{B}}$ on X_φ. This is a locally convex space whose topology $\tau_{X_\varphi^0,\mathcal{B}}$ is generated by the family $\{\|\cdot\|_{\alpha,\varphi}: \alpha \in \Delta\}$ of seminorms defined by

$$\|x\|_{\alpha,\varphi} = \sqrt{|\langle x, x\rangle_{\varphi,\mathcal{B}}|_\alpha}, \qquad x \in X_\varphi^0$$

One checks that $\|x\|_{\alpha,\varphi}^2 = |b^*\langle x, x\rangle_{\varphi,\mathcal{B}} b|_\alpha$ for $x \in X_\varphi^0$ and $b \in R(\mathcal{B})$. So, as $(\mathcal{B}, \tau_\mathcal{B})$ is a locally convex partial *-algebra, it follows that the right module action of $R(\mathcal{B})$ on X_φ^0 is continuous. Hence, the triple $(X_\varphi^0, \langle\cdot,\cdot\rangle_\mathcal{B}, \tau_{X_\varphi^0\mathcal{B}})$ is a locally convex $(\mathcal{B}, \tau_\mathcal{B})$-module. This concludes the proof. ∎

Remark. We write $(X_\varphi, \tau_{X_\varphi,\mathcal{B}})$ for the $\tau_{X_\varphi^0\mathcal{B}}$-completion of X_φ^0.

Definition 3.3. A member φ of $CP(\mathcal{A}, \mathcal{B})$ will be called *invariant* if the following three properties hold:

(i) The linear span of $\lambda_\varphi(R(\mathcal{A}) \otimes \mathcal{B})$ is dense in X_φ.

(ii) $\varphi(a \circ b, c) = \varphi(b, a^\# \circ c)$, $\forall a \in \mathcal{A}$, $b, c \in R(\mathcal{A})$.

(iii) $\varphi(a^\# \circ b, c \circ d) = \varphi(b, (a \circ c) \circ d)$ $\forall b, d \in R(\mathcal{A})$ and $a, c \in \mathcal{A}$ with $a \in L(c)$.

Notation. Write $ICP(\mathcal{A}, \mathcal{B})$ for the set of all *invariant* members of $CP(\mathcal{A}, \mathcal{B})$.

Remark 3.4. 1. The following fact will be employed later.

Let $\varphi \in ICP(\mathcal{A}, \mathcal{B})$ and $\xi = \sum_{j=1}^n a_j \otimes b_j$ be a member of N_φ, with $a_j \in R(\mathcal{A})$ and $b_j \in \mathcal{B}$, $j = 1, 2, \ldots, n$. Then, for any $a \in \mathcal{A}$, $\xi_a = \sum_{j=1}^n a \circ a_j \otimes b_j$ also lies in $N\varphi$.

This is seen as follows. Let $c_k \in R(\mathcal{A})$, $d_k \in \mathcal{B}$, $k = 1, 2, \ldots, m$, and $\eta = \sum_{k=1}^m c_k \otimes d_k$. Then, a simple calculation shows that

$$\langle\lambda_\varphi(\eta), \lambda_\varphi(\xi_a)\rangle_{\varphi,\mathcal{B}} = \langle\lambda_\varphi(\eta_{a^\#}), \lambda_\varphi(\xi)\rangle_{\varphi,\mathcal{B}} = 0$$

and as the linear span of $\lambda_\varphi(\mathcal{A} \otimes \mathcal{B})$ is dense in X_φ, it follows that $\lambda_\varphi(\xi_a) = 0$, implying $\xi_a \in N_\varphi$.

2. Our main result is the following.

Theorem 3.5. Let $(\mathcal{A}, {}^\#, \circ)$ be a unital partial *-algebra with unit $e_\mathcal{A}$; $(\mathcal{B}, \tau_\mathcal{B})$ a unital semiassociative, locally convex partial *-algebra with unit $e_\mathcal{B}$, involution *, and partial multiplication written as juxtaposition; and φ a member of $ICP(\mathcal{A}, \mathcal{B})$. Then, there are a locally convex $(\mathcal{B}, \tau_\mathcal{B})$-module $(D_\varphi, \langle\cdot,\cdot\rangle_\mathcal{B}, \tau_{D_\varphi,\mathcal{B}})$, a module *-representation π_φ of $(\mathcal{A}, {}^\#, \circ)$ in $L_{\text{mod}}^+(D_\varphi, X_\varphi)$, where X_φ is the $\tau_{D_\varphi,\mathcal{B}}$-completion of D_φ, and a linear map V_φ from $\mathcal{B}$ into D_φ such that

$$\langle\langle b_1, \varphi(x, y) b_2\rangle\rangle = \langle\pi_\varphi(x) V_\varphi b_1, \pi_\varphi(y) V_\varphi b_2\rangle_{\varphi,\mathcal{B}} \qquad (**)$$

for all $x, y \in \mathcal{A}$ and $b_1, b_2 \in R(\mathcal{B})$, and the linear span of $\pi_\varphi(R(\mathcal{A}))V_\varphi\mathcal{B}$ is dense in X_φ.

Remark. Denote $V_\varphi e_{\mathcal{B}}$ by ζ_φ. Observe that if $b_1 = e_{\mathcal{B}} = b_2$, then (**) reduces to the following:

$$\varphi(x, y) = \langle \pi_\varphi(x)\zeta_\varphi, \pi_\varphi(y)\zeta_\varphi \rangle_{\varphi,\mathcal{B}}$$

for all $x, y \in \mathcal{A}$.

Proof. Let $\varphi \in ICP(\mathcal{A}, \mathcal{B})$. Take λ_φ, N_φ, $\langle \cdot,\cdot \rangle_{\varphi,\mathcal{B}}$, and X_φ as in Proposition 3.2. Define D_φ as the linear span of $\lambda_\varphi(R(\mathcal{A}) \otimes \mathcal{B})$ and $\tau_{D_\varphi,\mathcal{B}}$ as the locally convex topology on D_φ generated by the family $\{ \|\cdot\|_{\alpha,D_\varphi}: \alpha \in \Delta\}$ of seminorms given by

$$\|x\|_{\alpha,D_\varphi} = \sqrt{|\langle x, x\rangle_{\varphi,\mathcal{B}}|_\alpha}, \qquad x \in D_\varphi$$

Then the triple $(D_\varphi, \langle \cdot,\cdot \rangle_{\mathcal{B}}, \tau_{D_\varphi,\mathcal{B}})$ is a locally convex $(\mathcal{B}, \tau_{\mathcal{B}})$-module. By Definition 3.3, Property (i), D_φ is dense in X_φ.

For each $a \in \mathcal{A}$, define $\pi_\varphi(a)$ on D_φ by

$$\pi_\varphi(a)\lambda_\varphi(\xi) = \lambda_\varphi(\xi_a)$$

with $\xi = \sum_{j=1}^n a_j \otimes b_j$, $\xi_a = \sum_{j=1}^n a \circ a_j \otimes b_j$, $a_1, \ldots, a_n \in R(\mathcal{A})$, $b_1, \ldots, b_n \in R(\mathcal{B})$. As $\xi_a \in N_\varphi$ whenever $\xi \in N_\varphi$ (by Remark 3.4), it follows that $\pi_\varphi(a)$ is well defined for each $a \in \mathcal{A}$. On D_φ, one checks that:

- $\pi_\varphi(\lambda_1 a_1 + \lambda_2 a_2) = \lambda_1\pi_\varphi(a_1) + \lambda_2\pi_\varphi(a_2)$ for all $a_1, a_2 \in \mathcal{A}$, $\lambda_1, \lambda_2 \in \mathbb{C}$.
- $\pi_\varphi(a)^+ = \pi_\varphi(a^\#)$, $\forall a \in \mathcal{A}$.
- $\pi_\varphi(a)$ acts linearly for each $a \in \mathcal{A}$.

It follows that $\pi_\varphi(a)$, $a \in \mathcal{A}$, extends to a linear map, denoted again by $\pi_\varphi(a)$, lying in $L^+_{\text{mod}}(D_\varphi, X_\varphi)$. Hence, by Proposition 2.4, $\pi_\varphi(a)$ is a module map for each $a \in \mathcal{A}$. Next, suppose $a, b \in \mathcal{A}$ with $a \in L(b)$ and $\xi, \eta \in R(\mathcal{A}) \otimes \mathcal{B}$. Then, using the invariance of φ, one finds

$$\langle \pi_\varphi(a^\#)\lambda_\varphi(\xi), \pi_\varphi(b)\lambda_\varphi(\eta)\rangle_{\varphi,\mathcal{B}} = \langle \lambda_\varphi(\xi), \pi_\varphi(a \circ b)\lambda_\varphi(\eta)\rangle_{\varphi,\mathcal{B}}$$

It follows that $\pi_\varphi(a) \in L(\pi_\varphi(b))$ whenever $a, b \in \mathcal{A}$ with $a \in L(\mathcal{A})$, showing that π_φ is a representation of $\mathcal{A}$ in $L^+_{\text{mod}}(D_\varphi, X_\varphi)$. As it has been seen above that π_φ is a *-map, this shows that π_φ is a module *-representation of $\mathcal{A}$ in $L^+_{\text{mod}}(D_\varphi, X_\varphi)$.

Define $V_\varphi: \mathcal{B} \to D_\varphi$ by

$$V_\varphi b = \lambda_\varphi(e_{\mathcal{A}} \otimes b)$$

for $b \in R(\mathcal{B})$. Then, V_φ is linear on $\mathcal{B}$. The linear span of $\pi_\varphi(R(\mathcal{A}))V_\varphi\mathcal{B}$ is precisely the linear span of $\lambda_\varphi(R(\mathcal{A}) \otimes \mathcal{B})$, which is D_φ, and is therefore dense in X_φ. Furthermore, for $b_1, b_2 \in R(\mathcal{B})$ and $x, y \in \mathcal{A}$, we have

$$\begin{aligned}\langle \pi_\varphi(x)V_\varphi b_1, \pi_\varphi(y)V_\varphi b_2\rangle_{\varphi,\mathcal{B}} &= \langle \lambda_\varphi(x \otimes b_1), \lambda_\varphi(y \otimes B_2)\rangle_{\varphi,\mathcal{B}} \\ &= b_1^*\varphi(x, y)b_2 \\ &= \langle\langle b_1, \varphi(x, y)b_2\rangle\rangle\end{aligned}$$

as claimed. This concludes the proof. ∎

Remark. Theorem 3.5 generalizes the well-known Stinespring characterization (Stinespring, 1955) of completely positive maps on C^*-algebras, as well as its recent extensions by Powers (1974), Lassner and Lassner (1977), Paschke (1973), and Ekhaguere and Odiobala (1991).

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